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On the solvability of a Neumann boundary value problem for the differential equation $f(t,x,x',x'')=0$

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Abstract

Using barrier strip arguments, we investigate the existence of $C^2[0,1]$-solutions to the Neumann boundary value problem $f(t,x,x',x'')=0$, $x'(0)=a$, $x'(1)=b$.

MSC: 34B15

Keywords: boundary value problem; equation unsolved with respect to the second derivative; Neumann boundary conditions; existence

1 Introduction

The purpose of this paper is to establish the existence of $C^2[0,1]$-solutions to the scalar Neumann boundary value problem (BVP)

$$\begin{cases} f(t,x,x',x'')=0, & t\in[0,1],\\ x'(0)=a, \quad x'(1)=b, & a\neq b, \end{cases} \tag{N}$$

where the function $f(t,x,p,q)$ and its first derivatives are continuous only on suitable subsets of the set $[0,1]\times R^3$.

The literature devoted to the solvability of singular and nonsingular Neumann BVPs for second order ordinary differential equations whose main nonlinearities do not depend on the second derivative is vast. We quote here only [1–5] for results and references.

The solvability of the homogeneous Neumann problem for the equation $(p(t)x')'+f(t,x,x',x'')=y(t)$, under appropriate conditions on f, has been studied in [6–8]. Results, concerning the existence of solutions to the homogeneous and nonhomogeneous Neumann problem for the equation $x''=f(t,x,x',x'')-y(t)$ can be found in [9] and [10] respectively. BVPs for the same equation with various linear boundary conditions have been studied in [9, 11–13]. The results of [14] guarantee the solvability of BVPs for the equation $x''=f(t,x,x',x'')$ with fully linear boundary conditions. BVPs for the equation $f(t,x,x',x'')=0$ with fully nonlinear boundary conditions have been studied in [15]. For results, which guarantee the solvability of the Dirichlet BVP for the same equation, in the scalar and in the vector cases, see [12] and [16] respectively.

Concerning the kind of the nonlinearity of the function $f(t,x,p,q)$, we note that it is assumed sublinear in [6], semilinear in [11] and linear with respect to x, p and q in [8, 12].

Finally, in [9] and [17] f is a linear function with respect to q, while with respect to p, it is a quadratic function or satisfies Nagumo type growth conditions respectively.

As in [10, 15, 18, 19], we use sign conditions to establish a priori bounds for x, x' and x'', where $x(t) \in C^2[0,1]$ is a solution to a suitable family of BVPs similar to that in [10, 19]. Using these a priori bounds and applying the topological transversality theorem from [20], we prove our main existence result.

2 Basic hypotheses

To formulate our hypotheses, we use the sets

$$J_x = \left[\min\left\{0, \frac{a+b}{2}, \frac{a^2}{2(a-b)}\right\}, \max\left\{0, \frac{a+b}{2}, \frac{a^2}{2(a-b)}\right\}\right] \quad \text{and}$$
$$J_p = \left[\min\{a,b\}, \max\{a,b\}\right].$$

So, we assume that there are positive constants K, M and a sufficiently small $\varepsilon > 0$ such that:

H1.

$$f(t,x,p,q) \text{ is continuous with respect to } x \in R \text{ for each } (t,p,q) \in [0,1] \times R^2,$$
$$f(t,x,p,q) \text{ is continuous with respect to } q \in R \text{ for each } (t,x,p) \in [0,1] \times R^2,$$

there are constants K_x and K_q such that

$$f_x(t,x,p,q) \geq K_x > 0 \quad \text{for } (t,x,p,q) \in [0,1] \times R^3,$$
$$f_q(t,x,p,q) \leq -K_q < 0 \quad \text{for } (t,x,p,q) \in [0,1] \times [-M_0 - \varepsilon, M_0 + \varepsilon] \times R^2,$$

where

$$M_0 = \max\left\{\frac{e}{e^2-1}\big(|a-be| + |ae-b|\big), \frac{L}{\min\{K, K_x, K_q\}} + \max\left\{\frac{|a+b|}{2}, \frac{a^2}{2|a-b|}\right\}\right\},$$

$f(t,x,p,b-a-(1-\lambda)x)$ is bounded for $(\lambda,t,x,p) \in [0,1]^2 \times J_x \times J_p$ and $L = \max\{\sup|f(t,x,p,b-a-(1-\lambda)x)|, \max K|b-a-(1-\lambda)x|\}$ for $(\lambda,t,x,p) \in [0,1]^2 \times J_x \times J_p$.

H2.

$$f(t,x,p,q) + Kq \geq 0 \quad \text{for } (t,x,p,q) \in [0,1] \times [-M_0 - \varepsilon, M_0 + \varepsilon] \times R \times (-\infty, -M)$$

and

$$f(t,x,p,q) + Kq \leq 0 \quad \text{for } (t,x,p,q) \in [0,1] \times [-M_0 - \varepsilon, M_0 + \varepsilon] \times R \times (M, \infty),$$

where M_0 is as in H1.

H3. The functions $f(t,x,p,q)$ and $f_q(t,x,p,q)$ are continuous for $(t,x,p,q) \in [0,1] \times [-M_0 - \varepsilon, M_0 + \varepsilon] \times [-M_1 - \varepsilon, M_1 + \varepsilon] \times [-M_2 - \varepsilon, M_2 + \varepsilon]$, where $M_1 = \min\{|a|,|b|\} + M_0 + M$, $M_2 = M_0 + M$ and M_0 is as in H1.

3 Auxiliary lemmas

In order to obtain our main existence results, we use the constant K from the hypotheses to construct the family of BVPs

$$\begin{cases} K(x'' - (1-\lambda)x) = \lambda(K(x'' - (1-\lambda)x) + f(t,x,x',(x''-(1-\lambda)x))), \\ x'(0) = a, \qquad x'(1) = b, \end{cases} \tag{3.1_λ}$$

where $\lambda \in [0,1]$ and prove the following three auxiliary results.

Lemma 3.1 *Let H1 hold and $x(t) \in C^2[0,1]$ be a solution to $(3.1)_\lambda$, $\lambda \in [0,1]$. Then*

$$|x(t)| \le M_0, \quad t \in [0,1].$$

Proof For $\lambda = 0$, problem $(3.1)_0$ is of the form

$$x'' - x = 0, \qquad x'(0) = a, \qquad x'(1) = b.$$

The unique solution to this BVP satisfies the bound

$$|x(t)| \le \frac{e}{e^2-1}\big(|a - be| + |ae - b|\big), \quad t \in [0,1].$$

Let now $\lambda \in (0,1]$. Then the function

$$y(t) = x(t) - s(t), \quad t \in [0,1], \text{ where } s(t) = \frac{1}{2}(b-a)t^2 + at, t \in [0,1],$$

is a solution to the homogeneous boundary value problem

$$\begin{aligned} &K\big(y'' + b - a - (1-\lambda)(y+s)\big) \\ &\quad = \lambda\big(K\big(y'' + b - a - (1-\lambda)(y+s)\big) + f\big(t, y+s, y'+s', y'' + b - a - (1-\lambda)(y+s)\big)\big), \\ &y'(0) = 0, \qquad y'(1) = 0. \end{aligned}$$

The equation is equivalent to the following one

$$\begin{aligned} &(1-\lambda)Ky'' \\ &\quad = (1-\lambda)^2 Ky - (1-\lambda)K\big(b - a - (1-\lambda)s\big) \\ &\qquad + \lambda f\big(t, y+s, y'+s', y'' + b - a - (1-\lambda)(y+s)\big) \\ &\qquad - \lambda f\big(t, s, y'+s', y'' + b - a - (1-\lambda)(y+s)\big) \\ &\qquad + \lambda f\big(t, s, y'+s', y'' + b - a - (1-\lambda)(y+s)\big). \end{aligned}$$

Hence, by the intermediate value theorem, we obtain consecutively

$$\begin{aligned} &(1-\lambda)Ky'' \\ &\quad = (1-\lambda)^2 Ky - (1-\lambda)K\big(b - a - (1-\lambda)s\big) \end{aligned}$$

$$\begin{aligned}
&+ \lambda f_x\bigl(t, s+\theta_1 y, y'+s', y''+b-a-(1-\lambda)(y+s)\bigr)y \\
&+ \lambda f\bigl(t, s, y'+s', y''+b-a-(1-\lambda)(y+s)\bigr) \\
&- \lambda f\bigl(t, s, y'+s', y''+b-a-(1-\lambda)s\bigr) \\
&+ \lambda f\bigl(t, s, y'+s', y''+b-a-(1-\lambda)s\bigr),
\end{aligned}$$

for any $\theta_1 \in (0,1)$ depending on $\lambda \in [0,1]$, $t \in [0,1]$ and $y(t)$,

$$\begin{aligned}
&(1-\lambda)Ky'' \\
&\quad = (1-\lambda)^2 Ky - (1-\lambda)K\bigl(b-a-(1-\lambda)s\bigr) \\
&\qquad + \lambda f_x\bigl(t, s+\theta_1 y, y'+s', y''+b-a-(1-\lambda)(y+s)\bigr)y \\
&\qquad + \lambda f_q\bigl(t, s, y'+s', y''+b-a-(1-\lambda)s-\theta_2(1-\lambda)y\bigr)\bigl((\lambda-1)y\bigr) \\
&\qquad + \lambda f\bigl(t, s, y'+s', y''+b-a-(1-\lambda)s\bigr) - \lambda f\bigl(t, s, y'+s', b-a-(1-\lambda)s\bigr) \\
&\qquad + \lambda f\bigl(t, s, y'+s', b-a-(1-\lambda)s\bigr)
\end{aligned}$$

and

$$\begin{aligned}
&(1-\lambda)Ky'' \\
&\quad = (1-\lambda)^2 Ky - (1-\lambda)K\bigl(b-a-(1-\lambda)s\bigr) \\
&\qquad + \lambda f_x\bigl(t, s+\theta_1 y, y'+s', y''+b-a-(1-\lambda)(y+s)\bigr)y \\
&\qquad + \lambda f_q\bigl(t, s, y'+s', y''+b-a-(1-\lambda)s-\theta_2(1-\lambda)y\bigr)\bigl(-(1-\lambda)y\bigr) \\
&\qquad + \lambda f_q\bigl(t, s, y'+s', b-a-(1-\lambda)s+\theta_3 y''\bigr)y'' + \lambda f\bigl(t, s, y'+s', b-a-(1-\lambda)s\bigr),
\end{aligned}$$

for any $\theta_2, \theta_3 \in (0,1)$ depending on $\lambda \in [0,1]$, $t \in [0,1]$ and $y(t)$,

$$\begin{cases}
((1-\lambda)K - \lambda f_q(t, s, y'+s', b-a-(1-\lambda)s+\theta_3 y''))y'' \\
\quad = \{(1-\lambda)^2 K + \lambda f_x(t, s+\theta_1 y, y'+s', y''+b-a-(1-\lambda)(y+s)) \\
\qquad - \lambda(1-\lambda) f_q(t, s, y'+s', y''+b-a-(1-\lambda)s-\theta_2(1-\lambda)y)\}y \\
\qquad + \lambda f(t, s, y'+s', b-a-(1-\lambda)s) - (1-\lambda)K(b-a-(1-\lambda)s).
\end{cases} \tag{3.2}$$

Next, suppose that $|y(t)|$ achieves its maximum at $t_0 \in (0,1)$. Then the function $z = y^2(t)$ has also a maximum at t_0. Consequently, we have

$$0 \geq z''(t_0) = 2y(t_0)y''(t_0). \tag{3.3}$$

Using the fact that $y'(t_0) = 0$, from (3.2) we obtain

$$\begin{cases}
\{(1-\lambda)K - \lambda f_q(t_0, s_0, s_0', b-a-(1-\lambda)s_0+\theta_{3,0}y_0'')\}y_0'' \\
\quad = \{(1-\lambda)\{(1-\lambda)K - \lambda f_q(t_0, s_0, s_0', y_0''+b-a-(1-\lambda)s_0-\theta_{2,0}(1-\lambda)y_0)\} \\
\qquad + \lambda f_x(t_0, s_0+\theta_{1,0}y_0, s_0', y_0''+b-a-(1-\lambda)(y_0+s_0))\}y_0 \\
\qquad + \lambda f(t_0, s_0, s_0', b-a-(1-\lambda)s_0) - (1-\lambda)K(b-a-(1-\lambda)s_0),
\end{cases} \tag{3.4}$$

where $\theta_{1,0} = \theta_1(t_0, s_0', y_0'' + b - a - (1-\lambda)(y_0 + s_0))$, $\theta_{2,0} = \theta_2(t_0, s_0, s_0')$, $\theta_{3,0} = \theta_3(t_0, s_0, s_0')$, and $s_0 = s(t_0)$, $s_0' = s'(t_0)$, $y_0 = y(t_0)$, $y_0'' = y''(t_0)$.

In view of H1, from (3.4) we have

$$\begin{cases} (1-\lambda)\{(1-\lambda)K - \lambda\bar{f}_q\} + \lambda\bar{f}_x \geq \min\{(1-\lambda)K - \lambda\bar{f}_q, \bar{f}_x\} \\ \qquad\qquad\qquad\qquad\qquad\qquad \geq \min\{K, -\bar{f}_q, \bar{f}_x\} \geq \min\{K, K_x, K_q\}, \end{cases} \tag{3.5}$$

where

$$\bar{f}_q = f_q\bigl(t_0, s_0, s_0', y_0'' + b - a - (1-\lambda)s_0 - \theta_{2,0}(1-\lambda)y_0\bigr),$$
$$\bar{f}_x = f_x\bigl(t_0, s_0 + \theta_{1,0}y_0, s_0', y_0'' + b - a - (1-\lambda)(y_0 + s_0)\bigr).$$

Suppose now that $|y(t_0)| > L(\min\{K, K_x, K_q\})^{-1}$. Then, from (3.4) and (3.5) it follows that

$$\begin{cases} \{(1-\lambda)K - \lambda f_q(t_0, s_0, s_0', b - a - (1-\lambda)s_0 + \theta_{3,0}y_0'')\}y_0'' \\ \quad \geq \min\{K, K_x, K_q\}y(t_0) \\ \qquad + \lambda f(t_0, s_0, s_0', b - a - (1-\lambda)s_0) - (1-\lambda)K(b - a - (1-\lambda)s_0) \end{cases} \tag{3.6}$$

if $y(t_0) > L(\min\{K, K_x, K_q\})^{-1}$ or

$$\begin{cases} \{(1-\lambda)K - \lambda f_q(t_0, s_0, s_0', b - a - (1-\lambda)s_0 + \theta_{3,0}y_0'')\}y_0'' \\ \quad \leq \min\{K, K_x, K_q\}y(t_0) \\ \qquad + \lambda f(t_0, s_0, s_0', b - a - (1-\lambda)s_0) - (1-\lambda)K(b - a - (1-\lambda)s_0) \end{cases} \tag{3.7}$$

if $y(t_0) < -L(\min\{K, K_x, K_q\})^{-1}$. Multiplying (3.6) and (3.7) by $y(t_0)$, we obtain

$$\bigl\{(1-\lambda)K - \lambda f_q\bigl(t_0, s_0, s_0', b - a - (1-\lambda)s_0 + \theta_{3,0}y_0''\bigr)\bigr\}y_0''y_0$$
$$\quad \geq y_0\bigl(\min\{K, K_x, K_q\}y_0 - L\bigr) > 0,$$
$$\bigl\{(1-\lambda)K - \lambda f_q\bigl(t_0, s_0, s_0', b - a - (1-\lambda)s_0 + \theta_{3,0}y_0''\bigr)\bigr\}y_0''y_0$$
$$\quad \geq |y_0|\bigl(\min\{K, K_x, K_q\}|y_0| - L\bigr) > 0,$$

respectively. Finally, since $(t_0, s_0, s_0', b - a - (1-\lambda)s_0 + \theta_{3,0}y_0'') \in [0,1] \times [-M_0 - \varepsilon, M_0 + \varepsilon] \times R^2$ we have $f_q(t_0, s_0, s_0', b - a - (1-\lambda)s_0 + \theta_{3,0}y_0'') < 0$. So

$$y_0''y_0 > 0,$$

which contradicts (3.3). Thus, we infer that if $|y(t)|$ achieves its maximum on $(0,1)$, then

$$\bigl|y(t)\bigr| \leq \frac{L}{\min\{K, K_x, K_q\}} \quad \text{for } t \in [0,1] \text{ and } \lambda \in (0,1].$$

Let $|y(1)|$ be the maximum of $|y(t)|$ and suppose that $|y(1)| > L(\min\{K, K_x, K_q\})^{-1}$. Following the above reasoning and using the fact that $y'(1) = 0$, we obtain

$$y(1)y''(1) > 0.$$

If $y(1) > 0$, then $y''(1) > 0$ and so $y'(t)$ is a strictly increasing function for $t \in U_1$, where $U_1 \subset [0,1]$ is a sufficiently small neighbourhood of $t = 1$. So, we see that

$$y'(t) < y'(1) = 0 \quad \text{for } t \in U_1 \setminus \{1\},$$

i.e., $y(t)$ is a strictly decreasing function for $t \in U_1$. Therefore, $y(1) = |y(1)|$ can not be the maximum of $|y(t)|$ on $[0,1]$, which is a contradiction. Assume next that $y(1) < 0$. Then similar to the above arguments lead again to a contradiction. Thus, we see that

$$\left|y(1)\right| \leq \frac{L}{\min\{K, K_x, K_q\}}.$$

The inequality

$$\left|y(0)\right| \leq \frac{L}{\min\{K, K_x, K_q\}}$$

can be obtained in the same manner. Consequently, the eventual solutions of $(3.1)_\lambda$, $\lambda \in (0,1]$ satisfy the bound

$$\left|x(t)\right| \leq \left|y(t)\right| + \left|s(t)\right| \leq \frac{L}{\min\{K, K_x, K_q\}} + \max\left\{\frac{a^2}{2|a-b|}, \frac{|a+b|}{2}\right\}, \quad t \in [0,1],$$

and the proof of the lemma is completed. □

Lemma 3.2 *Let H1 and H2 hold and $x(t) \in C^2[0,1]$ be a solution to $(3.1)_\lambda$, $\lambda \in [0,1]$. Then:*
(a) $|x''(t) - (1-\lambda)x(t)| \leq M$, $|x''(t)| \leq M_0 + M$, $t \in [0,1]$.
(b) $|x'(t)| \leq \min\{|a|,|b|\} + M_0 + M$, $t \in [0,1]$.

Proof (a) Suppose there exists a $(\lambda_0, t_0) \in [0,1]^2$ or a $(\lambda_1, t_1) \in [0,1]^2$ such that

$$x''(t_0) - (1-\lambda_0)x(t_0) < -M \quad \text{or} \quad x''(t_1) - (1-\lambda_1)x(t_1) > M.$$

By Lemma 3.1, we have

$$\left|x(t)\right| \leq M_0 \quad \text{for } t \in [0,1]. \tag{3.8}$$

In particular, (3.8) holds for t_0 and t_1. Thus, in view of H2, we have

$$\begin{aligned} 0 &> K\big(x''(t_0) - (1-\lambda_0)x(t_0)\big) \\ &= \lambda_0\big\{K\big(x''(t_0) - (1-\lambda_0)x(t_0)\big) + f\big(t_0, x(t_0), x'(t_0), \big(x''(t_0) - (1-\lambda_0)x(t_0)\big)\big)\big\} \geq 0 \end{aligned}$$

or

$$\begin{aligned} 0 &< K\big(x''(t_1) - (1-\lambda_1)x(t_1)\big) \\ &= \lambda_1\big\{K\big(x''(t_1) - (1-\lambda_1)x(t_1)\big) + f\big(t_1, x(t_1), x'(t_1), \big(x''(t_1) - (1-\lambda_1)x(t_1)\big)\big)\big\} \leq 0, \end{aligned}$$

respectively. The obtained contradictions show that

$$-M \le x''(t) - (1-\lambda)x(t) \le M \quad \text{for } t \in [0,1] \text{ and } \lambda \in [0,1],$$

and therefore

$$-(M_0 + M) \le x''(t) \le M_0 + M \quad \text{for } t \in [0,1],$$

which proves (a).

(b) By the mean value theorem, for each $t \in (0,1]$ there is a $\xi \in (0,t)$ such that

$$x'(t) - x'(0) = x''(\xi)t.$$

Since, in view of (a), we have $|x''(\xi)| \le M_0 + M$, from the last formula we find that

$$\left|x'(t)\right| \le \left|x'(0)\right| + \left|x''(\xi)\right| \le \min\left\{|a|, |b|\right\} + M_0 + M, \quad t \in [0,1],$$

which proves (b) and completes the proof of the lemma. □

Lemma 3.3 *Let H1, H2 and H3 hold. Then there exists a function $G(\lambda, t, x, p)$ continuous for $(\lambda, t, x, p) \in [0,1]^2 \times [-M_0 - \varepsilon, M_0 + \varepsilon] \times [-M_1 - \varepsilon, M_1 + \varepsilon]$ and such that*

(a) the BVP

$$x'' - (1-\lambda)x = G\left(\lambda, t, x, x'\right), \quad t \in [0,1],$$
$$x'(0) = a, \qquad x'(1) = b,$$

is equivalent to BVP $(3.1)_\lambda$.

(b) $G(0, t, x, p) = 0$ for $(t, x, p) \in \Pi_q \equiv [0,1] \times [-M_0 - \varepsilon, M_0 + \varepsilon] \times [-M_1 - \varepsilon, M_1 + \varepsilon]$.

Proof (a) We write the differential equation from $(3.1)_\lambda$ as

$$\lambda f\left(t, x, x', \left(x'' - (1-\lambda)x\right)\right) - (1-\lambda)K\left(x'' - (1-\lambda)x\right) = 0 \tag{3.9}$$

and consider the function

$$F(\lambda, t, x, p, q) = \lambda f(t, x, p, q) - (1-\lambda)Kq \quad \text{for } (\lambda, t, x, p, q) \in [0,1] \times \Pi,$$

where $\Pi = [0,1] \times [-M_0 - \varepsilon, M_0 + \varepsilon] \times [-M_1 - \varepsilon, M_1 + \varepsilon] \times [-M_2 - \varepsilon, M_2 + \varepsilon]$. Since $-M_2 - \varepsilon < -M$ and $M_2 + \varepsilon > M$, we can use H2 to conclude that

$$F(\lambda, t, x, p, -M_2 - \varepsilon)F(\lambda, t, x, p, M_2 + \varepsilon) < 0 \quad \text{for } (\lambda, t, x, p) \in [0,1] \times \Pi_q. \tag{3.10}$$

On the other hand, for $(\lambda, t, x, p, q) \in [0,1] \times \Pi$ we have

$$F_q(\lambda, t, x, p, q) = \lambda f_q(t, x, p, q) - (1-\lambda)K \le \max\{-K, -K_q\} < 0. \tag{3.11}$$

Finally, from H3 we have that

$$F(\lambda,t,x,p,q) \text{ and } F_q(\lambda,t,x,p,q) \text{ are continuous for } (\lambda,t,x,p,q) \in [0,1] \times \Pi. \tag{3.12}$$

So, (3.10), (3.11) and (3.12) allow us to apply a well-known theorem to conclude that there is a unique function $G(\lambda,t,x,p)$ which is continuous for $(\lambda,t,x,p) \in [0,1] \times \Pi_q$ and such that the equations

$$q = G(\lambda,t,x,p), \quad (\lambda,t,x,p) \in [0,1] \times \Pi_q$$

and

$$F(\lambda,t,x,p,q) = 0, \quad (\lambda,t,x,p,q) \in [0,1] \times \Pi$$

are equivalent. Now from Lemma 3.1 we have

$$-M_0 - \varepsilon \le x(t) \le M_0 + \varepsilon \quad \text{for } t \in [0,1],$$

and Lemma 3.2 yields

$$-M_1 - \varepsilon \le x'(t) \le M_1 + \varepsilon \quad \text{and} \quad -M_2 - \varepsilon < -M \le x''(t) - (1-\lambda)x(t) \le M < M_2 + \varepsilon$$

for $t \in [0,1]$ and $\lambda \in [0,1]$. Consequently, equation (3.9) is equivalent to the equation

$$x'' - (1-\lambda)x = G\big(\lambda,t,x,x'\big), \quad t \in [0,1],$$

which yields the first assertion.

(b) It follows immediately from $F(0,t,x,p,0) = 0$ for $(t,x,p) \in \Pi_q$. □

4 The main result

Our main result is the following existence theorem, the proof of which is based on the lemmas of the previous sections and the Topological transversality theorem [20].

Theorem 4.1 *Let H1, H2 and H3 hold. Then problem (N) has at least one solution in* $C^2[0,1]$.

Proof First, we observe that according to Lemma 3.3, the family of boundary value problems

$$\begin{cases} x'' - (2-\lambda)x = G(\lambda,t,x,x') - x, & t \in [0,1], \\ x'(0) = a, \qquad x'(1) = b, \end{cases} \tag*{$(3.13)_\lambda$}$$

is equivalent to the family $(3.1)_\lambda$ for $\lambda \in [0,1]$. Next define the set

$$U = \big\{x \in C_B^2[0,1] : |x| < M_0 + \varepsilon, \big|x'\big| < M_1 + \varepsilon, \big|x''\big| < M_2 + \varepsilon\big\},$$

where $C_B^2[0,1] = \{x(t) \in C^2[0,1] : x'(0) = a, x'(1) = b\}$, and the maps

$$j : C_B^2[0,1] \to C^1[0,1] \quad \text{by } jx = x,$$
$$G_\lambda : C^1[0,1] \to C[0,1] \quad \text{by } (G_\lambda x)(t) = G\big(\lambda, t, x(t), x'(t)\big) - x(t),$$

where $t \in [0,1]$, $\lambda \in [0,1]$, $x(t) \in j(\overline{U})$ and

$$L_\lambda : C_B^2[0,1] \to C[0,1] \quad \text{by } L_\lambda x = x'' - (2-\lambda)x, \lambda \in [0,1].$$

Since L_λ, $\lambda \in [0,1]$, is a continuous, linear, one-to-one map of $C_B^2[0,1]$ onto $C[0,1]$, the map L_λ^{-1}, $\lambda \in [0,1]$ exists and is continuous. In addition, G_λ, $\lambda \in [0,1]$, is a continuous and j is a completely continuous embedding. Since $j(\overline{U})$ is a compact subset of $C^1[0,1]$, and G_λ, $\lambda \in [0,1]$, and L_λ^{-1}, $\lambda \in [0,1]$, are continuous on $j(\overline{U})$ and $G_\lambda(j(\overline{U}))$ respectively, the homotopy

$$H : \overline{U} \times [0,1] \to C^2[0,1] \quad \text{defined by } H(x,\lambda) \equiv H_\lambda(x) \equiv L_\lambda^{-1} G_\lambda j(x)$$

is compact. Besides, the equation

$$L_\lambda^{-1} G_\lambda j(x) = x \quad \text{for } x \in \overline{U} \quad \text{yields} \quad L_\lambda x = G_\lambda j(x),$$

which coincides with BVP $(3.13)_\lambda$. Thus, the fixed points of $H_\lambda(x)$ are solutions to $(3.13)_\lambda$. But, from Lemma 3.1 and Lemma 3.2 it follows that the solutions to $(3.13)_\lambda$ are elements of U. Consequently, $H_\lambda(x)$, $\lambda \in [0,1]$, is a fixed point free on ∂U, *i.e.*, $H_\lambda(x)$ is an admissible map for all $\lambda \in [0,1]$. Finally, we see that the map H_0 is a constant map, *i.e.*, $H_0(x) \equiv l$, where l is the unique solution to the BVP

$$x'' - 2x = -x, \qquad x'(0) = a, \qquad x'(1) = b.$$

From the fact that $l \in U$, it follows that H_0 is an essential map (see, [20]). By the Topological transversality theorem (see, [20]), $H_1 = L_1^{-1} G_1 j$ is also essential, *i.e.*, problem $(3.13)_1$ has a $C^2[0,1]$-solution. It is also a solution to $(3.1)_1$, by Lemma 3.3. To complete the proof, remark that problem $(3.1)_1$ coincides with the problem (N). □

We conclude with the following example, which illustrates our main result.

Example 4.2 Consider the boundary value problem

$$1 - (1.5 + t)x'' - tx''^5 - \cos x' + x = 0,$$
$$x'(0) = 0, \qquad x'(1) = 10^{-4}.$$

It is clear that for $(t,x,p,q) \in [0,1] \times R^3$ the function

$$f(t,x,p,q) = 1 - (1.5+t)q - tq^5 - \cos p + x$$

is continuous and $f_x(t,x,p,q) = 1$ and $f_q(t,x,p,q) = -1.5 - t - 5tq^4$. Thus H1 holds for $K_x = 1$ and $K_q = 1.5$.

To verify H2 we choose, for example, $K = 0.5$, $M = 5$ and $\varepsilon = 3 \cdot 10^{-5}$. Next we need the constants L and M_0. Having in mind that $J_x = [0, 5 \cdot 10^{-5}]$ and $J_p = [0, 10^{-4}]$, from

$$5 \cdot 10^{-5} \le 10^{-4} - (1-\lambda)x \le 10^{-4} \quad \text{for } (\lambda, x) \in [0,1] \times J_x$$

it follows that $\max K|b - a - (1-\lambda)x| = 0.5 \max(10^{-4} - (1-\lambda)x) = 5 \cdot 10^{-5}$. On the other hand, from

$$-2,5 \cdot 10^{-4} - 10^{-20} \le -(1,5+t)\big(10^{-4} - (1-\lambda)x\big) - t\big(10^{-4} - (1-\lambda)x\big)^5 \le -7,5 \cdot 10^{-5}$$

for $(\lambda, t, x) \in [0,1]^2 \times J_x$ and

$$0 \le 1 - \cos p \le 5 \cdot 10^{-9} \quad \text{for } p \in J_p$$

we have

$$\begin{aligned} -26 \cdot 10^{-5} &< 1 - (1,5+t)\big(10^{-4} - (1-\lambda)x\big) - t\big(10^{-4} - (1-\lambda)x\big)^5 - \cos p + x \\ &\le -2,5 \cdot 10^{-5} + 5 \cdot 10^{-9} \end{aligned}$$

for $(\lambda, t, x, p) \in [0,1]^2 \times J_x \times J_p$, which means that for $(\lambda, t, x, p) \in [0,1]^2 \times J_x \times J_p$

$$\begin{aligned} &\max\big|f\big(t, x, p, b - a - (1-\lambda)x\big)\big| = \\ &\quad = \max\big|1 - (1,5+t)\big(10^{-4} - (1-\lambda)x\big) - t\big(10^{-4} - (1-\lambda)x\big)^5 - \cos p + x\big| \le 26 \cdot 10^{-5}. \end{aligned}$$

So, $L = \max\{26 \cdot 10^{-5}, 5 \cdot 10^{-5}\} = 26 \cdot 10^{-5}$. Then

$$M_0 = \max\left\{\frac{e}{e^2 - 1}\big(10^{-4}e + 10^{-4}\big), \frac{26 \cdot 10^{-5}}{\min\{0.5, 1, 1.5\}} + 5 \cdot 10^{-5}\right\} = 57 \cdot 10^{-5}$$

and we see that for $(t, x, p, q) \in [0,1] \times [-M_0 - \varepsilon, M_0 + \varepsilon] \times R \times (-\infty, -M)$

$$f(t, x, p, q) + Kq = -(1+t)q - tq^5 + 1 - \cos p + x > 0$$

and

$$f(t, x, p, q) + Kq < 0 \quad \text{for } (t, x, p, q) \in [0,1] \times [-M_0 - \varepsilon, M_0 + \varepsilon] \times R \times (M, \infty).$$

Thus, H2 also holds.

Finally, H3 holds since $f(t, x, p, q)$ and $f_q(t, x, p, q)$ are continuous for $(t, x, p, q) \in [0,1] \times R^3$.

Thus, we can apply Theorem 4.1 to conclude that the considered problem has a solution in $C^2[0,1]$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors declare that the study was realized in collaboration with the same engagement.

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